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Evidence for the Poisson Distribution for Quasi-Energies In the Quantum Kicked-Rotator Model

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A transformation on the two-dimensional torus which is related to the problem of limit distribution for the distance between the levels in the kicked-rotator model is considered. The first four moments of the r.w. which describe the numbers of visits of a point in a rectangle of measure ε are calculated. It is shown that when $\varepsilon \to 0$ they converge to the first four moments of a Poisson r.w.

KEY WORDS: Torus; measure preserving transformation; moments; Poisson r.w.

SECTION 1

In this note we study a transformation on the torus which is related to the problem of limit distribution for the distance between the levels in the kicked-rotator model.[1]

We denote by Tor^2 the two dimensional torus (which we identify with the square $[0, 1] \times [0, 1]$ after the identification of the opposite sides) and by $dxdy$ the Lebesgue measure on it.

On Tor^2 we define the following transformation T:

$$T(x, y) = (x + \alpha, x + y)$$

where $\alpha \in [0, 1]$ is an irrational number.

It is easy to see that the measure $dxdy$ is T invariant. It is easy to see also that this transformation is not mixing and has zero entropy.

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We can iterate the transformation and obtain the following formula (see Ref. (2)).

$$T^n(x, y) = \left(x + n\alpha,\ y + nx + \frac{n(n-1)}{2}\alpha \right) \tag{1.1}$$

For every $\varepsilon > 0$, we define the region

$$\prod_{\varepsilon} = \{(x, y) \in \mathrm{Tor}^2 \colon 0 \leqslant x \leqslant 1\ 0 \leqslant y \leqslant \varepsilon\}$$

and define the random variable

$$\zeta^{\alpha,\varepsilon}(x, y) = \#\left\{k \colon T^k(x, y) \in \prod_{\varepsilon} 0 \leqslant k < \frac{c}{\varepsilon}\right\}$$

($\#\{\cdot\}$ = cardinality of the set $\{\cdot\}$) where c is a positive constant.

Our goal would be to show that $\zeta^{\alpha,\varepsilon}(x, y)$ has in the limit as $\varepsilon \to 0$, a Poisson distribution with parameter c. This is apparently a difficult problem. We make a step towards its solution by evaluating the moments of $\zeta^{\alpha,\varepsilon}(x, y)$ up to the fourth order. These moments depend on α and this poses further problems. For this reason we consider an averaging procedure by performing an integration over α. In this way we obtain a simpler expression for the n moment of the $\zeta^{\alpha,\varepsilon}(x, y)$. We will show that the n moment, with $n \leqslant 4$, converges to the same moment of a Poisson r.w. with the parameter c. For clearness we write the first four moments of the Poisson distribution r.w. with parameter c. They have the form c, $c + c^2$, $c + 3c^2 + c^3$, $c + 7c^2 + 6c^3 + c^4$.

SECTION 2

In this section we derive the explicit formula for the n moment, after the integration over α.

The object that we want to study is the quantity

$$\int_0^1 d\alpha \int_0^1 \int_0^1 dx\, dy (\zeta^{\alpha,\varepsilon}(x, y))^n \tag{2.1}$$

For this purpose we introduce the following variables

$$\xi_k^{\alpha,\varepsilon}(x, y) = \begin{cases} 1 & \text{if} \quad T^k(x, y) \in \prod_\varepsilon 0 \leqslant k < c/\varepsilon \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

and obviously

$$\zeta^{\alpha,\varepsilon}(x, y) = \sum_{0 \leqslant k < c/\varepsilon} \xi_k^{\alpha,\varepsilon}(x, y) \tag{2.3}$$

We denote

$$\xi_0^{\alpha,\varepsilon}(x, y) \equiv \xi_0^{\varepsilon}(x, y)$$

Now we prove the following.

Proposition 1.1.

$$(2.1) =$$

$$= \varepsilon^n \sum_{\substack{k_1,\ldots,k_n \\ 0 \leqslant k_j < c/\varepsilon}} \sum_{\substack{m_1,\ldots,m_n \\ m_1+m_2+\cdots+m_n=0 \\ m_1k_1+\cdots+m_nk_n=0 \\ m_1k_1^2+\cdots+m_nk_n^2=0}} \prod_{j=1}^{n} h(\varepsilon m_j) \tag{2.4}$$

where

$$h(t) = \frac{\exp(2\pi i t) - 1}{2\pi i t}$$

Proof. Taking the Fourier series for the function $\xi_l^{\alpha,\varepsilon}$ and using the invariance of the Lebesgue measure on Tor^2 we obtain

$$\xi_l^{\alpha,\varepsilon}(x, y) = \sum_{\substack{m_1,m_2 \\ m_2 l - m_1 = 0}} \exp[2\pi i(m_1 x + m_2 y)] \times \exp\left[2\pi i\left(m_1 l\alpha - m_2 \frac{l(l+1)}{2}\right)\right] \varepsilon h(-\varepsilon m_2) \tag{2.5}$$

Now (2.1) can be written as (we omit in the sum the condition $0 \leqslant k_i < c/\varepsilon$)

$$\sum_{k_1,\ldots,k_n} \int_0^1 d\alpha \int_0^1 \int_0^1 dx\, dy\, \xi_{k_1}^{\alpha,\varepsilon}(x, y) \cdots \xi_{k_n}^{\alpha,\varepsilon}(x, y)$$

$$= \sum_{k_1,\ldots,k_n} \int_0^1 d\alpha \int_0^1 \int_0^1 dx\, dy\, \xi_0^{\varepsilon}(x, y)\, \xi_{k_2-k_1}^{\alpha,\varepsilon}(x, y) \cdots \xi_{k_n-k_1}^{\alpha,\varepsilon}(x, y)$$

$$= \sum_{k_1,\ldots,k_n} \sum_{\substack{m_1,m_2,\ldots,m_{2n-2} \\ m_2(k_2-k_1)-m_1=0 \\ m_4(k_3-k_1)-m_3=0 \\ \vdots \\ m_{2n-2}(k_n-k_1)-m_{2n-3}=0}} \int_0^1 d\alpha$$

$$\times \exp\left[2\pi i\left(m_1(k_2-k_1) - m_2\frac{(k_2-k_1)(k_2-k_1+1)}{2}\right)\alpha\right]$$

$$\times \exp\left[2\pi i\left(m_3(k_3-k_1)-m_4\frac{(k_3-k_1)(k_3-k_1+1)}{2}+\cdots\right.\right.$$

$$\left.\left.+m_{2(n-1)-1}(k_n-k_1)-m_{2(n-1)}\frac{(k_n-k_1)(k_n-k_1+1)}{2}\right)\alpha\right]$$

$$\times \int_0^1\int_0^1 dx\, dy \exp[2\pi i(m_1+m_3+\cdots+m_{2(n-1)-1})\, x$$

$$+2\pi i(m_2+m_4+\cdots+m_{2(n-1)})\, y$$

$$\times \xi_0^\varepsilon(x,\, y)\, \varepsilon^{n-1}h(-\varepsilon m_2)\cdots h(-\varepsilon m_{2(n-1)})$$

$$= \sum_{k_1,\ldots,k_n} \sum_{\substack{m_2,\ldots,m_{2(n-1)} \\ m_1+m_3+\cdots+m_{2(n-1)-1}=0 \\ m_2(k_2+k_1)-m_1=0 \\ m_4(k_3-k_1)-m_3=0 \\ \vdots \\ m_{2(n-1)}(k_n-k_1)-m_{2(n-1)-1}=0 \\ m_1(k_2-k_1)-m_2\,(k_2-k_1)(k_2-k_1+1)/2+\cdots \\ +m_{2(n-1)-1}(k_n-k_1)-m_{2(n-1)}(k_n-k_1)(k_n-k_1+1)/2=0}} \varepsilon^n h(-\varepsilon m_2)\cdots$$

$$\times h(-\varepsilon m_{2(n-1)})\, h(\varepsilon(m_2+m_4+\cdots+m_{2(n-1)}))$$

From the last formula taking $m'_1=-m_2,\cdots, m'_n=-m_{2(n-1)}$ and $m'_n=m_2+m_4+\cdots+m_{2(n-1)}$ and changing the order of ks we obtain (2.4).

SECTION 3

In this section we evaluate the n moment for $n\leqslant 4$.

The first moment is easily evaluated via the invariance of the measure and is equal to c.

The second one is given by

$$\sum_{\substack{k_1,k_2 \\ 0\leqslant k_l<c/\varepsilon}} \sum_{\substack{m_1,m_2 \\ m_1+m_2=0 \\ m_1k_1+m_2k_2=0 \\ m_1k_1^2+m_2k_2^2=0}} \varepsilon^2 h(\varepsilon m_1)\, h(\varepsilon m_2)$$

$$= c^2+\varepsilon \sum_{0\leqslant k<c/\varepsilon} \varepsilon \sum_{m\neq 0} \frac{\exp(2\pi i\varepsilon m)-1}{2\pi i\varepsilon m}\,\frac{\exp\ -2\pi i\varepsilon m-1}{-2\pi i\varepsilon m}$$

$$=c^2+c\varepsilon \sum_{m\neq 0} 2\,\frac{(1-\cos(2\pi\varepsilon m))}{(2\pi\varepsilon m)^2}$$

$$= c^2+c\varepsilon \sum_{m\neq 0} \frac{\sin^2(2\pi\varepsilon m/2)^2}{(2\pi\varepsilon m/2)^2}$$

the last sum in the limit $\varepsilon \to 0$ converges to the integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin^2(x/2)}{(x/2)^2}\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2}\,dx = 1 \tag{3.1}$$

So we get for the second moment $c^2 + c$.

The third moment is equal to

$$\sum_{\substack{k_1,k_2,k_3\\ 0\leqslant k_i < c/\varepsilon}} \sum_{\substack{m_1,m_2,m_3\\ m_1+m_2+m_3=0\\ m_1k_1+m_2k_2+m_3k_3=0\\ m_1k_1^2+m_2k_2^2+m_3k_3^2=0}} \varepsilon^3 h(\varepsilon m_1)\, h(\varepsilon m_2)\, h(\varepsilon m_3)$$

$$= \varepsilon^3 \sum_{\substack{k_1,k_2,k_3\\ 0\leqslant k_i < c/\varepsilon}} + 3\varepsilon^3 \sum_{\substack{k_1,k_2,k_3\\ 0\leqslant k_i < c/\varepsilon}} \sum_{\substack{m_1,m_2\\ m_1+m_2=0,\, m_1k_1+m_2k_2=0\\ m_1k_1^2+m_2k_2^2=0}} h(\varepsilon m_1)\, h(\varepsilon m_2)$$

$$+ \varepsilon^3 \sum_{\substack{k_1,k_2,k_3\\ 0\leqslant k_i < c/\varepsilon}} \sum_{\substack{m_1,m_2,m_3\\ m_i\neq 0\\ m_1+m_2+m_3=0\\ m_1k_1+m_2k_2+m_3k_3=0\\ m_1k_1^2+m_2k_2^2+m_3k_3^2=0}} h(\varepsilon m_1)\, h(\varepsilon m_2)\, h(\varepsilon m_3) \tag{3.2}$$

The first and the second terms in the r.h.s. of (3.2) are evaluated as before. The first gives c^3 while the limit of the second one is equal to $3c^2$.

The third one is equal to

$$\varepsilon^3 \sum_{0\leqslant k < c/\varepsilon} \sum_{m_1,m_2} h(\varepsilon m_1)\, h(\varepsilon m_2)\, h(-\varepsilon(m_1+m_2)) \tag{3.3}$$

(3.3) follows from the fact that the only solution of the system

$$m_1 + m_2 + m_3 = 0$$
$$m_1k_1 + m_2k_2 + m_3k_3 = 0$$
$$m_1k_1^2 + m_2k_2^2 + m_3k_3^2 = 0$$

is $k_1 = k_2 = k_3$. This easily can be obtained by direct computations. Now (3.3) becomes

$$c\varepsilon^2 \sum_{m_1,m_2} 2\,\frac{\sin(2\pi\varepsilon m_1) + \sin(2\pi\varepsilon m_2) - \sin(2\pi\varepsilon(m_1+m_2))}{2\pi\varepsilon m_1\, 2\pi\varepsilon m_2\, 2\pi\varepsilon(m_1+m_2)}$$

and in the limit $\varepsilon \to 0$ we obtain

$$c\,\frac{1}{2\pi^2}\int_{R^2} dx\,dy\,\frac{\sin x + \sin y - \sin(x+y)}{xy(x+y)}$$

where the integral is equal to $2\pi^2$, and so we obtain the result for the third moment.

For the fourth moment we obtain from (2.4)

$$\varepsilon^4 \sum_{\substack{k_1,k_2,k_3,k_4 \\ 0 \leqslant k_i < c/\varepsilon}} \sum_{\substack{m_1,m_2,m_3,m_4 \\ \sum m_i = 0,\ \sum m_i k_i = 0,\ \sum m_i k_i^2 = 0}} h(\varepsilon m_1)\, h(\varepsilon m_2)\, h(\varepsilon m_3)\, h(\varepsilon m_4)$$

$$= \varepsilon^4 \sum_{\substack{k_1,k_2,k_3,k_4 \\ 0 \leqslant k_i < c/\varepsilon}} + 4c \left(\varepsilon^3 \sum_{\substack{k_1,k_2,k_3 \\ 0 \leqslant k_i < c/\varepsilon}} \sum_{\substack{m_1,m_2,m_3 \\ m_i \neq 0 \\ \sum m_i = 0,\ \sum m_i k_i = 0,\ \sum m_i k_i^2 = 0}} \right.$$

$$+ 6c^2 \left(\varepsilon^2 \sum_{\substack{k_1,k_2 \\ 0 \leqslant k_i < c/\varepsilon}} \sum_{\substack{m_1,m_2 \\ m_i \neq 0 \\ m_1 + m_2 = 0 \\ m_1 k_1 + m_2 k_2 = 0 \\ m_1 k_1^2 + m_2 k_2^2 = 0}} h(\varepsilon m_1)\, h(\varepsilon m_2) \right)$$

$$+ \varepsilon^4 \sum_{\substack{k_1,k_2,k_3,k_4 \\ 0 \leqslant k_i < c/\varepsilon}} \sum_{\substack{m_1,m_2,m_3,m_4 \\ m_i \neq 0 \\ \sum m_i = 0,\ \sum m_i k_i = 0,\ \sum m_i k_i^2 = 0}} h(\varepsilon m_1)\, h(\varepsilon m_2)\, h(\varepsilon m_3)\, h(\varepsilon m_4) \tag{3.4}$$

From what we have seen above it is clear that the problem is to calculate only the last term in the r.h.s. of (3.4). In order to do this we have to study the diophantine equations

$$\sum_{i=1}^{4} m_i = 0, \ \sum_{i=1}^{4} m_i k_i = 0, \ \sum_{i=1}^{4} m_i k_i^2 = 0 \tag{3.5}$$

in the variables k_1, k_2, k_3, k_4. Now from (3.5) we obtain the system

$$\begin{cases} m_1 k_1 + m_2 k_2 + m_3 k_3 = (m_1 + m_2 + m_3)\, k_4 \\ m_1 k_1^2 + m_2 k_2^2 + m_3 k_3^2 = (m_1 + m_2 + m_3)\, k_4^2 \end{cases}$$

Denoting

$$a_1 = (m_1 + m_2 + m_3)\, k_4$$

and

$$a_2 = (m_1 + m_2 + m_3)\, k_4^2$$

we want to solve the system

$$\begin{cases} m_1 k_1 + m_2 k_2 + m_3 k_3 = a_1 \\ m_1 k_1^2 + m_2 k_2^2 + m_3 k_3^2 = a_2 \end{cases} \tag{3.6}$$

From the first equation of the system (3.6) we can find k_3 and using the second one we get

$$m_1 k_1^2 + m_2 k_2^2 + m_3 \frac{(a_1 - m_1 k_1 - m_2 k_2)^2}{m_3^2} = a_2$$

After simple calculations we obtain the following quadratic form in k_1 and k_2.

$$\begin{aligned} m_1(m_1+m_2)\, k_1^2 + m_2(m_2+m_3)\, k_2^2 + 2m_1 m_2 k_1 k_2 - 2a_1 m_1 k_1 \\ - 2a_1 m_2 k_2 + a_1^2 - m_2 a_2 = 0 \end{aligned} \tag{3.7}$$

We shall use the classical Gauss' theory of integer quadratic forms.[3] We are interested in integer solutions of (3.7). For this reason we must study the determinant of (3.7). It is equal to

$$\Delta = \begin{vmatrix} m_1(m_1+m_3) & m_1 m_2 & -a_1 m_1 \\ m_1 m_2 & m_2(m_2+m_3) & -a_1 m_2 \\ -a_1 m_1 & -a_1 m_2 & a_1^2 - m_3 a \end{vmatrix}$$

$$= m_1 m_2 m_3^2 (a_1^2 - (m_1 + m_2 + m_3)\, a_2) = 0$$

Introducing the quantities

$$-f = \begin{vmatrix} m_1 m_2 & -a_1 m_1 \\ m_2(m_2+m_3) & -a_1 m_2 \end{vmatrix} = a_1 m_1 m_2 m_3 \tag{3.8a}$$

$$-e = -\begin{vmatrix} m_1(m_1+m_3) & -a_1 m_1 \\ m_1 m_2 & -a_1 m_2 \end{vmatrix} = a_1 m_1 m_2 m_3 \tag{3.8b}$$

$$-d = \begin{vmatrix} m_1(m_1+m_3) & m_1 m_2 \\ m_1 m_2 & m_2(m_2+m_3) \end{vmatrix} = m_1 m_2 m_3 (m_1 + m_2 + m_3) \tag{3.8c}$$

and making the change of variables

$$x = dk_1 - f \quad y = dk_2 - e$$

we obtain

$$m_1(m_1+m_3)\, x^2 + 2m_1 m_2 xy + m_2(m_2+m_3)\, y^2 = 0 \tag{3.9}$$

(3.9) can be written in general (i.e. $m_1 + m_3 \neq 0$ or $m_2 + m_3 \neq 0$, the cases in which they can be zero will treated below) as

$$\left(x - \frac{-m_1 m_2 + \sqrt{d}}{m_1(m_1+m_3)}\, y\right)\left(x - \frac{-m_1 m_2 - \sqrt{d}}{m_1(m_1+m_3)}\, y\right) = 0 \tag{3.10}$$

Now if $d>0$ but is not a square number or $d<0$ we have only one integer solution of (3.10): $x=y=0$. Coming back to the ks variables we have

$$k_1=\frac{f}{d}\, k_2=\frac{e}{d}$$

and recalling the definition of f and e we have $k_1=k_2$. Now we can easily see that

$$k_1=\frac{-a_1 m_1 m_2 m_3}{-m_1 m_2 m_3(m_1+m_2+m_3)}=k_4$$

and from the last equality it follows $k_1=k_2=k_3=k_4$. So the contribution of this term is equal to

$$\varepsilon^4 \sum_{\substack{k\\ 0\leqslant k<c/\varepsilon}} \sum_{\substack{m_1,m_2,m_3,m_4\\ m_i\neq 0\\ \sum m_i=0}} h(\varepsilon m_1)\, h(\varepsilon m_2)\, h(\varepsilon m_3)\, h(\varepsilon m_4) \tag{3.11}$$

Now we must analyse the case when d is a square. This can happen only in the following five cases: $m_1=-m_2, m_1=-m_3, m_2=-m_3$, $m_1=-m_2=m_3$, $m_1=-m_3=m_2$.

The typical situation for the first three cases is

$$\varepsilon^4 \sum_{\substack{k_1,k_2,k_3,k_4\\ 0\leqslant k_i<c/\varepsilon}} \sum_{\substack{m_1,m_2\\ m_3=-m_1,m_4=-m_2\\ m_1(k_1-k_3)+m_2(k_2-k_4)=0\\ m_1(k_1^2-k_3^2)+m_2(k_2^2-k_4^2)=0}} h(\varepsilon m_1)\, h(-\varepsilon m_1)\, h(\varepsilon m_2)\, h(-\varepsilon m_2) \tag{3.12}$$

In order to evaluate (3.12) we must solve the system of equations

$$\begin{cases} m_1(k_1-k_3)+m_2(k_2-k_4)=0\\ m_1(k_1^2-k_3^2)+m_2(k_2^2-k_4^2)=0 \end{cases} \tag{3.13}$$

There are two solutions of the sysem (3.13): one is $k_2=k_4$ and $k_1=k_3$, the other is

$$\begin{aligned} k_1&=\frac{m_1-m_2}{m_1+m_2}k_3+\frac{2m_2}{m_1+m_2}k_4\\ k_2&=\frac{2m_1}{m_1+m_2}k_3+\frac{m_2-m_1}{m_1+m_2}k_4 \end{aligned} \tag{3.14}$$

We are interested only in integer positive solutions of (3.14). Because of this we must take m_1 and m_2 of the following kind

$$m_1 = (1-p)\,r \quad m_2 = (1+p)\,r$$

where $p \neq -1, 0, 1$, $r \neq 0$ are integer numbers.

In this way (3.14) becomes

$$k_1 = -pk_3 + (1+p)\,k_4$$

$$k_2 = (1-p)\,k_3 + pk_4$$

From (3.12) we have that the contribution of this case is given by

$$\Bigg| \varepsilon^4 \sum_{\substack{k_1,k_2,k_3,k_4 \\ 0 \leqslant k_i < c/\varepsilon}} \sum_{\substack{p,r \\ p \neq -1,0,1\ r \neq 0 \\ k_1 = -pk_3 + (1+p)\,k_4,\, k_2 = (1-p)\,k_3 + pk_4}} h(\varepsilon(1-p)\,r)$$

$$\times h(\varepsilon(1-p)\,r)\, h(-\varepsilon(1-p)\,r)\, h(\varepsilon(1+p)\,r)\, h(-\varepsilon(1+p)\,r) \Bigg|$$

$$\leqslant \varepsilon^2 \sum_{\substack{k_3,k_4 \\ 0 \leqslant k_i < c/\varepsilon}} \varepsilon \sum_{p,r} \frac{\sin^2\left(\dfrac{2\pi\varepsilon^{1/2}(1-p)\,\varepsilon^{1/2}r}{2}\right)}{\left(\dfrac{2\pi\varepsilon^{1/2}(1-p)\,\varepsilon^{1/2}r}{2}\right)^2} \frac{\sin^2\left(\dfrac{2\pi\varepsilon^{1/2}(1+p)\,\varepsilon^{1/2}r}{2}\right)}{\left(\dfrac{2\pi\varepsilon^{1/2}(1+p)\,\varepsilon^{1/2}r}{2}\right)^2}$$

and the quantity inside the modulus, as $\varepsilon \to 0$ goes to

$$\frac{1}{2\pi} \int_{R^2} dx\, dy \frac{\sin^4\left(\dfrac{xy}{2}\right)}{\left(\dfrac{xy}{2}\right)^4}$$

This shows that this contribution tends to zero, so that the only contribution to (3.12) is from $k_1 = k_3$, $k_2 = k_4$. Obviously we have only three such cases. The other two cases where d is a square i.e. $m_1 = -m_2 = m_3 = -m_4$, $m_1 = -m_3 = m_2 = -m_4$ give a zero contribution too, as it is easy to see.

The final formula, in which we omit the terms that do not contribute in the limit as $\varepsilon \to 0$, is l.h.s. of (3.4)

$$= \varepsilon^4 \sum_{\substack{k_1,k_2,k_3,k_4 \\ 0 \leqslant k_i < c/\varepsilon}} + 4c\left(\varepsilon^3 \sum_{\substack{k_1,k_2,k_3 \\ 0 \leqslant k_i < c/\varepsilon}} \sum_{\substack{m_1,m_2,m_3 \\ \sum m_i = 0,\ \sum m_i k_i = 0,\ \sum m_i k_i^2 = 0}} h(\varepsilon m_1) h(\varepsilon m_2) h(\varepsilon m_3)\right)$$

$$+ 6c^2\left(\varepsilon^2 \sum_{\substack{k_1,k_2 \\ 0 \leqslant k_i < c/\varepsilon}} \sum_{\substack{m_1,m_2 \\ m_i \neq 0 \\ m_1+m_2=0,\ m_1k_1+m_2k_2=0 \\ m_1k_1^2+m_2k_2^2=0}} h(\varepsilon m_1)\, h(\varepsilon m_2)\right)$$

$$+ \varepsilon^4 \sum_{\substack{k \\ 0 \leqslant k < c/\varepsilon}} \sum_{\substack{m_1,m_2,m_3,m_4 \\ m_i \neq 0 \\ \sum m_i = 0}} h(\varepsilon m_1)\, h(\varepsilon m_2)\, h(\varepsilon m_3)\, h(\varepsilon m_4)$$

$$+ 3\varepsilon^4 \sum_{\substack{k_1,k_2 \\ 0 \leqslant k_i < c/\varepsilon}} \sum_{\substack{m_1,m_2 \\ m_1+m_2 \neq 0 \\ m_1=-m_3,\, m_2=-m_4}} h(\varepsilon m_1)\, h(-\varepsilon m_1)\, h(\varepsilon m_2)\, h(-\varepsilon m_2) + O(\varepsilon) \tag{3.15}$$

In the limit the r.h.s. of (3.15) is equal to

$$c^4 + 4c^2 + 6c^3 + 3c^2 + c\frac{1}{4\pi^3} \times \int_{R^3} \frac{\begin{pmatrix}\cos(x+y+z) - \cos(x+y) - \cos(x+z) \\ -\cos(y+z) + \cos x + \cos y + \cos z - 1\end{pmatrix}}{xyz(x+y+z)}\, dx\, dy\, dz$$

The integral in the last formula is equal to $4\pi^3$ and so we obtain the result.

Remark. It is possible to calculate the first and the second moment without integrating on α, and the result is the same.

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